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A graphical method for finding possible quantizable free-field theories for higher spin

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Abstract. Graphs associated with the Gel'fand–Yaglom equations for arbitrary spin are used in a simple method to find the number of conditions which a good quantizable free-field theory must satisfy. The number of conditions is seen to be simply related to the line independence numbers of the s block subgraphs of the theory. This provides a rough guide as to whether a particular theory is possibly quantizable. Some examples are given. Also a general result is given which enables quick elimination of a range of 'bad' theories by visual examination of their graphs.

1. Introduction

The usual approach to the study of higher-spin quantum fields and their interactions is to start with a good Lagrangian free-field theory (ie one which is consistently quantizable) and introduce interactions as additive terms to the free-field Lagrangian. So far, this approach has been largely confined to a few well known good free-field theories such as the Dirac–Fierz–Pauli spin $\frac{3}{2}$ and the various spin 2 theories. The difficulties of introducing interactions consistently in these theories are well known. It is natural to ask whether there are other good free-field theories which may be stable to interactions. This question was posed by Wightman (1968) who observed that covariant field equations which may be amenable to the stable introduction of interactions might be found among those Gel'fand-Yaglom equations having physically reasonable mass spectra. This is obviously not sufficient, for such equations must also be quantizable, and field propagation in interaction must be casual. However, since any covariant field theory can be obtained by basing the Gel'fand–Yaglom equations on a sufficiently complicated representation of the proper Lorentz group $\mathscr{L}_{\mathrm{p}}$ and choosing the coefficient matrices appropriately, it is important to study these equations, their mass–spin spectra and quantization, and look for possible good theories. Shamaly and Capri (1973) for example, have unified a wide range of spin 1 theories by considering the possible Gel'fand–Yaglom equations based on a particular general representation of $\mathscr{L}_{\mathrm{p}}$.

Gel'fand and Yaglom have made a thorough study of the classical field theory of their equations, summarized in the classic book by Gel'fand *et al* (1963). Independently the same equations were studied by Bhabha, Harish-Chandra and many others. Thus a great deal is known about the classical theory, for example how to calculate the mass–spin spectra, the conditions necessary for definite charge and energy etc. However, in practice it is still very difficult to find good Gel'fand–Yaglom free field theories, and until recently, little has been done on their quantization. In previous papers (Cox 1974a, b) the author has described a graphical approach to the Gel'fand–Yaglom theory which helps in

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finding good theories. A lattice graph is associated with the representation of $\mathscr{L}_p$ used in the theory and this graph reflects the structure of the coefficient matrices of the Gel'fand–Yaglom equations. The idea is to make statements about the theory by examination of the graph, which is of a very simple form. In this paper we give some further results in this approach, describing a quick heuristic method of finding possible good Gel'fand–Yaglom theories and a method of eliminating some bad ones. Having found a possible good theory, there is still a lot of work to verify it, but the technique at least provides a rough guide. It is particularly useful for eliminating bad theories.

We consider only integral spin and we do not allow repeated irreducible representations of $\mathscr{L}_p$ in the theory. Also mass–spin states are required to be non-degenerate and physical states of zero charge or energy are not allowed. The terminology used here is more in line with current graph theory practice and differs slightly from that of the previous papers.

In § 2 we outline the graphical representation of the Gel'fand–Yaglom theory, which has been given in detail elsewhere (Cox 1974b). In § 3 we specify the detailed conditions which a good free-field theory must satisfy, from the point of view of mass spectra and quantization.

In § 4 we show the connection between the number of conditions imposed by a specified mass spectrum and the line independence numbers of the s block graphs. The line independence number of a graph can be obtained by finding a maximal matching on the graph and in the case of s block graphs this is usually easy to do by inspection. Some examples are given in § 5, showing how to isolate possible good theories. The actual verification of the theories has been given elsewhere.

In § 6 we give a general result which is useful in eliminating a wide range of representations of $\mathscr{L}_p$ from consideration as good theories. Section 7 is the conclusion.

2. Graphical representation of the theory

We use the notation of Cox (1974b). We plot the irreducible representations τ_i, of $\mathscr{L}_p$ as points in the (l_0, l_1) plane. Those points corresponding to finite representations occupy the fan $l_i > |l_0|$ in the upper half plane, where for integer spin, both l_0 and l_1 are both integers. Any finite-dimensional field theory without repeated representations will correspond to a finite subset of points in this region. The Gel'fand–Yaglom equation is

$$(L_\mu \partial^\mu + \mathrm{i}\chi)\psi = 0 \tag{2.1}$$

where ψ transforms according to a reducible representation $\mathscr{R}$ of $\mathscr{L}$. In the 'canonical' basis $\mathscr{L}_0$ has the form

$$L_0 = [C_s^{\tau\tau'} \delta_{ss'} \delta_{mm'}] \tag{2.2}$$

and so is completely reducible into 's blocks'

$$A_s = [C_s^{\tau\tau'}] \tag{2.3}$$

whose 'elements' are in fact scalar matrices. The spin index s takes the values $s = |l_0|, |l_0|+1, \ldots l_1 - 1$ for the irreducible representation $\tau \sim (l_0, l_1)$, while the m index takes the value $m = -s, -(s-1), \ldots, s-1, s$. The m index is always suppressed in the s blocks, so that in fact the dimension of A_s is given by the number of representations $(l_0^{(i)}, l_1^{(i)})$ such that $|l_0^{(i)}| \leqslant s \leqslant l_1^{(i)} - 1$.

The existence of a real invariant Lagrangian from which (2.1) is derivable imposes further conditions on the $C_s^{\tau\tau'}$ elements. In particular the $C_s^{\tau\tau'}$ are non-zero only for 'linked' representation τ, τ' such that

$$(l'_0, l'_1) = (l_0 \pm 1, l_1) \qquad \text{or} \qquad (l'_0, l'_1) = (l_0, l_1 \pm 1). \tag{2.4}$$

We associate a linear graph G with a particular Gel'fand–Yaglom theory by plotting as vertices the irreducible representations τ_i contained in $\mathscr{R}$ and connecting with an (undirected) edge those vertices corresponding to representations satisfying (2.4). The result will be some finite subgraph of a square lattice graph, and this graph will characterize the general Gel'fand–Yaglom theory based on that particular representation $\mathscr{R}$, in that it will reflect the block matrix structure of L_0.

The representations τ_i which appear in a particular s block A_s will be those corresponding to vertices in or on the rectangle

$$l_0 = -s, \qquad l_0 = s$$
$$l_1 = s+1, \qquad l_1 = j+1$$

in the $l_1 > |l_0|$ fan, where j is the maximum value of s in the representation. This will correspond to a particular subgraph G_s of G, the graph of the theory. It is these subgraphs G_s, representing the s blocks of the theory, on which we wish to concentrate, for as shown in Cox (1974a) the entire theory may be discussed in terms of these s blocks. The mass–spin spectrum of the theory is decided by the non-zero eigenvalues of the s blocks; the zero eigenvalues of the s blocks determine the mathematical complexity of the theory ('subsidiary' and 'auxiliary' conditions), and therefore ultimately the sensitivity toward interactions, and the conditions for quantizability of the free-field theory may be expressed solely in terms of trace conditions on the s block.

We will often need to regard G_s as a directed graph (digraph), with an edge pointing in either direction between any two vertices, we will then denote it by D_s. The edges of D_s may be labelled with the corresponding elements of A_s, which may conversely be written down by reference to graph D_s.

3. Conditions imposed by a good theory

To see whether a given Gel'fand–Yaglom theory can support a particular mass–spin spectrum it is necessary to find the characteristic polynomial of each s block $\Delta_s(\lambda)$ and try to choose the coefficients, which will be functions of the $C_s^{\tau_i\tau_j}$, to give this mass spectrum. Even for small representations $\mathscr{R}$ this is a difficult algebraic problem, and it is desirable to look for methods which take advantage of the simple structure of the s blocks as reflected in their graphs. We have given (Cox 1974b) a graphical technique for finding the coefficients of the $\Delta_s(\lambda)$ as functions of the parameters $C_s^{\tau_i\tau_j}$. This consists of visually searching the graph D_s for sets of disjoint directed cycles of total length $2r$, from which we can construct the required coefficients. As a first step in the search for a good theory it is useful to know how many conditions we need to satisfy to give a particular quantizable theory. For all but one of the coefficients of $\Delta_s(\lambda)$ which do not vanish identically, there will be a condition imposed by the required mass–spin spectrum. It follows from the condition $A_s^\dagger\Lambda_s = \Lambda_s A_s$ where Λ_s is the restriction to the s subspace of the invariant hermitian form Λ used in constructing the Lagrangian, that these coefficients are real functions of the arbitrary parameters $C_s^{\tau\tau'}$. Note that this does not necessarily imply that the eigenvalues of A_s are real. Thus each non-vanishing coefficient gives a real condition. The mass–spin spectrum alone does not fully determine the theory. Possibly distinct theories with the same mass–spin spectra may be obtained by choosing s blocks with the same characteristic polynomials but distinct minimal polynomials. The minimal polynomials are important in the quantization and indirectly in the introduction of interactions since they partly determine the nature of the subsidiary conditions which are so troublesome in the field equations. The extra conditions implied by specifying particular minimal polynomials are difficult to count and so we assume the best possible case, where only the characteristic polynomial is specified, and the minimal polynomial left arbitrary. If there are r non-zero mass states in the theory, then there will be $r-1$ trace conditions for quantization (Cox 1974a), in addition to the conditions imposed on the $\Delta_s(\lambda)$ coefficients.

The number of arbitrary parameters $C_s^{\tau\tau'}$ available to satisfy the above conditions is easy to count from the graph G, remembering that from space reflection covariance G must be symmetrical about the l_1 axis. So to get a rough guide to whether a good theory is possible, we have only to find the number of non-vanishing coefficients of the $\Delta_s(\lambda)$. In the next section we derive a simple visual method for this, using the s block graphs G_s.

The general problem here is to use the simple structure of a graph to make statements about a matrix associated with it. This type of problem is receiving some attention from graph theorists at the moment, although their main concern is with the adjacency matrix of a graph (elements 0 or 1) (Wilson 1972). However, many of those results of this 'graph eigenvalue theory' relating to connectivity properties only, carry over to the more general case where the non-zero elements of the associated matrix can be arbitrary to some extent.

4. Non-vanishing coefficients of the $\Delta_s(\lambda)$

We first notice that G and G_s are bipartite (bigraphs). That is, the vertex set can be partitioned into two disjoint sets V_1, V_2 such that no vertex in V_1 (V_2) is connected to any other vertex in $V_1(V_2)$. This means that it is possible to number the vertices so that the s blocks take the simple form

$$A_s = \begin{bmatrix} 0 & C_s \\ B_s & 0 \end{bmatrix}$$

which in itself can be useful. A bigraph is characterized by the fact that all its cycles are even, and it is a consequence of this that the eigenvalues of any matrix associated with such a graph are 'paired', ie of the form $\pm\alpha$. From this follows the general form of the $\Delta_s(\lambda)$ given in Cox (1974a). We can write

$$\Delta_s(\lambda) = \sum_{r=0}^{n} C_{2r}^{(s)}(-\lambda)^{n-2r}.$$

$C_{2r}^{(s)}$ ($r \geqslant 1$) is the sum of all the terms corresponding to sets of disjoint direct cycles of total length $2r$ in D_s, and $C_0^{(s)} = 1$. It is a non-vanishing polynomial in the $C_s^{\tau_i\tau_j}$ if and only if there exists at least one such set of cycles. Since the existence of a set of disjoint cycles of total length $2r$ implies a set of total length $2s$ for all $s < r$ it is sufficient to find the maximum value of r, and this will give the total number of conditions implied by a specified mass spectrum. For some graphs $\max(r) = n/2$—in this case it is possible to find a set of disjoint cycles which covers all vertices of the graph, or all but one vertex. In general however this will not be possible and some of the $C_{2r}^{(s)}$ will be identically zero simply by virtue of the connectivity of the graph. To find $l = \max(r)$ we have only to find a set of disjoint directed cycles of D_s of maximum length. Since all the cycles of D_s are of even length and any two adjacent vertices define a cycle of length 2, it is always possible to replace any set of disjoint cycles by a set of disjoint 2-cycles covering the same vertices and with the same total length. But 2-cycles in D_s correspond to edges in the undirected graph G_s and so a set of disjoint 2-cycles of total length $2r$ in D_s corresponds to a set of r disjoint edges in G_s. So l will be equal to the maximum number of disjoint edges it is possible to have in the graph G_s.

If G is any graph, an *independent set of edges* of G has no two edges adjacent, and is called a *matching* of the graph (it identifies pairs of indices). The maximum cardinality of such a set is called the *line independence number* $\beta_1(G)$ of the graph, and any set of $\beta_1(G)$ independent lines is called a *maximal matching* of G. What we have demonstrated above is that for s block, A_s with associated graph G_s, the number of conditions imposed by a given mass spectrum is:

$$l_s = \beta_1(G_s).$$

The problem of finding maximal matchings on a bigraph is well known in graph theory and there exist straightforward algorithms for solving it (Berge 1973). For our graphs this is hardly necessary as a visual inspection of the graph is usually sufficient to find a maximal matching and hence l_s quickly. In the case when the graph is in fact a tree (no cycles) a simple algorithm for finding $\beta_1(G_s)$ is as follows: start at an isolated edge with vertices (v, u), v being the external vertex of degree 1, and u being the internal vertex. Remove from the graph the 'star' consisting of the edge (v, u) and all other edges adjacent to it. Repeat this procedure on the remaining graph, starting at another isolated edge and continue removing stars in this way until all that is left is a set of disjoint edges (possibly null). The number of stars removed and the number of edges remaining are added to give $\beta_1(G_s)$.

Having found l_s for each s block, we add these to get the total number of conditions implied by any given mass–spin spectrum. This, with the number of trace conditions imposed by quantization must then be compared with the number of arbitrary parameters $C_s^{\tau\tau'}$. If the number of conditions exceeds the number of parameters, then a good theory is very unlikely. If there are more parameters than conditions then a good theory may be possible, but this will of course depend on the nature of the conditions. Also, from the way the $C_s^{\tau\tau'}$ are combined in the $\Delta_s(\lambda)$ coefficients it becomes apparent that for a high likelihood of a good theory the number of parameters must exceed the number of conditions by a fair margin. The examples given in the next section will illustrate this.

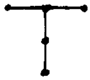
5. Some examples

5.1. Very high spin theories

As we have described elsewhere (Cox 1974b), for spin greater than 8, the number of conditions outweighs the number of arbitrary parameters available. It is therefore unlikely that good Gel'fand–Yaglom theories exist for such high spin (without repeated representations of $\mathscr{L}_\mathrm{p}$). To verify this algebraically would be difficult.

5.2. $\mathscr{R} = (0, 1) \oplus (0, 2) \oplus (0, 3) \oplus (-1, 3) \oplus (1, 3)$

The graph of $\mathscr{R}$ is

$G =$ 

and its s-block graphs are:

$G_0 =$  $G_1 =$  $G_2 =$

$\beta_1(G_0) = 1$ $\beta_1(G_1) = 1$ $\beta_1(G_2) = 1$

Thus there are 3 real conditions imposed by mass spectra. From the graph G, there are 3 arbitrary complex parameters to fulfil these conditions, so there seems to be here a possibility of a good theory. It is not difficult to verify that a good unique mass theory is in fact possible (Cox 1974b) although $\mathscr{R}$ will not sustain a multi-mass theory.

5.3. $\mathscr{R} = (0, 1) \oplus (0, 2) \oplus (0, 3) \oplus (-1, 2) \oplus (1, 2) \oplus (-2, 3) \oplus (-1, 3) \oplus (1, 3) \oplus (2, 3)$

The graph of $\mathscr{R}$ is

$G =$ 

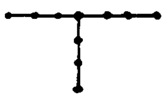
and its s block graphs are

$G_0 =$  $G_1 =$  $G_2 =$

$\beta_1(G_0) = 1$ $\beta_1(G_1) = 3$ $\beta_1(G_2) = 2$

So there are 6 real conditions imposed by mass spectra, and 6 arbitrary complex parameters with which to satisfy these. Again, a detailed analysis of the theory based on $\mathscr{R}$ reveals a wide range of possible good theories—including multi-mass theories.

5.4. $\mathscr{R} = (0, 1) \oplus (0, 2) \oplus (0, 3) \oplus (0, 4) \oplus (-3, 4) \oplus (-2, 4) \oplus (-1, 4) \oplus (1, 4) \oplus$
$(2, 4) \oplus (3, 4)$

$G =$ 

The s block graphs are

$G_0 =$  $G_1 =$  $G_2 =$  $G_3 =$

and $\beta_1(G_0) = \beta_1(G_1) = 2, \beta_1(G_2) = \beta_1(G_3) = 3.$

So there are 10 real conditions imposed by mass spectra, and $6 \times 2 = 12$ real parameters with which to satisfy these. The excess of parameters over conditions is not great in this case and in fact it is easy to see that the above graph cannot lead to a good theory. This fact is a consequence of a general result which we describe in the next section.

6. The elimination of certain types of bad theories

Amar and Dozzio (1972) have given the general form of graphs, for quantizable theories, given the minimum value l of s for which A_s is diagonalizable. Any good theory must be a subgraph of such graphs, and in the special case of unique mass theories Amar and Dozzio have shown that these graphs are necessary and sufficient. However, an important point is that it may not be possible to construct a theory from a given graph which satisfies the assumed conditions of their result—that is a theory having unique mass and certain blocks diagonalizable. This is the case for example with theories based on the representation

$$\mathscr{R} = \Sigma_{\oplus}(0, s+1).$$

The graph of this representation lies within the graph of Amar and Dozzio, but in general does not contain enough arbitrary parameters to satisfy the conditions required for unique mass (Shamaly and Capri 1971). We now give a result which can be used to eliminate a wide range of possible theories by a quick inspection of their graphs.

6.1. Result

Let G be the graph of a representation $\mathscr{R}$, with s block subgraphs D_s. Let l be the minimum value of s for which A_s is diagonalizable, so that G is in fact restricted to be a certain subgraph in the (l_0, l_1) plane as described by Amar and Dozzio. Then a quantizable theory with non-zero charge and energy density and non-degenerate mass–spin states is not possible if any D_s for $s < l$ contains exactly one set of disjoint cycles of total length $2[n/2]$ where A_s is $n \times n$.

By more careful wording the result can be strengthened, as will be seen from the proof, but as stated above it is sufficient to indicate the general idea.

6.2. Proof

If say D_j ($j < l$) has exactly one set of disjoint cycles of total length $2[n/2]$ then the coefficient $C^{(j)}_{2[n/2]}$ consists of a single term which is a product of factors corresponding to disjoint edges of G_s. This term cannot therefore be made to vanish without changing the graph G. However, since this term does not vanish, A_j has at most one zero eigenvalue, the rest being non-zero. Since theories referred to in the result cannot have s blocks with repeated non-zero eigenvalues, it follows that A_j must be diagonalizable, contrary to the assumption $j < l$.

Because of the symmetry of the Gel'fand–Yaglom graphs, it is not necessary that D_j has exactly one set of disjoint cycles of total length $2[n/2]$ to eliminate a good theory by the above result. It may have two sets, which due to symmetry contribute the same terms to $C^{(j)}_{2[n/2]}$—then diagonalizability will again be unavoidable.

Section 5.4 illustrates the above result. We have only to consider D_2, which clearly has exactly one set of disjoint cycles of total length 6. Unless we break the graph, the

2-block will therefore be diagonalizable, and by the work of Amar and Dozzio it cannot therefore be quantizable.

The above result becomes particularly powerful when looking for, or trying to eliminate unique mass theories. In this case, all but one s block must be nilpotent and if for any such s block with digraph D_s there exists exactly one set of disjoint cycles of total length $2r$ for any r, then we cannot have a good unique mass theory.

Variations and extensions of the above result can be used to eliminate many types of representation from providing good unique or multi-mass theories. The generalization of § 5.4 is as follows.

If any Gel'fand–Yaglom graph contains a D_s which is topologically isomorphic to a 'T-graph' such as shown in figure 1, with $2r$ (r even) horizontal edges and an odd number of vertical edges, then the corresponding A_s is diagonalizable and therefore the theory is not quantizable. Figure 2 shows a further example eliminating a possible spin 7

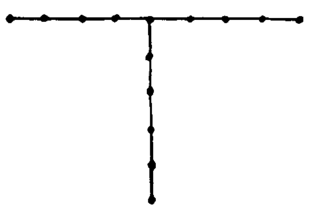


Figure 1. A typical T graph not leading to a good Gel'fand–Yaglom theory.

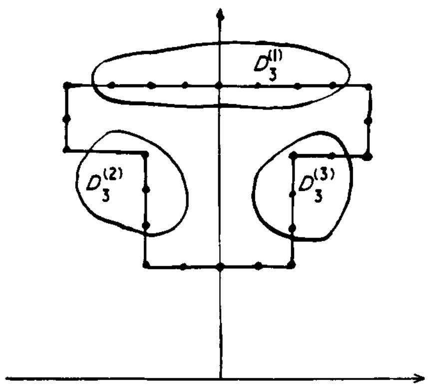


Figure 2. A spin seven theory with disconnected 3-block. Owing to the equivalence of $D_3^{(2)}$ and $D_3^{(3)}$ this graph cannot represent a good theory.

theory. The D_3 graph is disconnected, so A_3 will be completely reducible and its characteristic polynomial will be the product of those of $D_3^{(1)}$, $D_3^{(2)}$, $D_3^{(3)}$. Since by symmetry those of $D_3^{(2)}$ and $D_3^{(3)}$ are the same and repeated non-zero eigenvalues are not allowed, the submatrices corresponding to $D_3^{(2)}$ and $D_3^{(3)}$ must be taken as nilpotent. Since each is a chain of odd length, they fall to the general result above and must vanish altogether if they are to be nilpotent. So the graph of figure 2 cannot give a good theory. The extension of these ideas to similar and other graphs should now be obvious.

7. Conclusion

We have described a heuristic method for finding possible good Gel'fand–Yaglom theories and given a general result which allows us to rule out many types of graph as candidates for good theories. When a possible good theory has been found it must be checked and the actual representation of L_0 obtained, from which the Lagrangian may be determined. Frank (1973) has recently shown how to convert from the Gel'fand–Yaglom representation to the more usual spin-tensor formulation if this is desired, but it seems preferable to develop the complete quantum field theory for the Gel'fand–Yaglom equations, including the case of L_0 singular and non-diagonalizable. Much of this has already been done by Aurilla and Umezawa (1969), Schwinger (1951), Wightman and Capri (Wightman 1968).

It has still not been possible to prove categorically that all higher-spin interacting quantum field theories are inconsistent (in the sense of Velo and Zwanzinger 1969a, b, and Johnson and Sudarshan 1961) nor to actually exhibit a good higher-spin theory. In the absence of a general inconsistency proof we are still encouraged to look for good interaction theories. A first step in this search is to find good free-field theories and in this paper we have given an approach which helps in finding good free-field Gel'fand–Yaglom equations.

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